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Non abelian vortices as instantons on noncommutative discrete space

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Abstract: There seems to be close relationship between the moduli space of vortices and the moduli space of instantons, which is not yet clearly understood from a standpoint of the field theory. We clarify the reasons why many similarities are found in the methods for constructing the moduli of instanton and vortex, viewed in the light of the notion of the self-duality. We show that the non-Abelian vortex is nothing but the instanton in $R^2 \times Z_2$ from a viewpoint of the noncommutative differential geometry and the gauge theory in discrete space. The action for pure Yang-Mills theory in $R^2 \times Z_2$ is equivalent to that for Yang-Mills-Higgs theory in R^2.

Keywords: Solitons Monopoles and Instantons, Non-Commutative Geometry.

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1. Introduction

It is widely recognized that the exact solutions of field equations play an important role in analyzing the properties of the field theory even in the framework of a quantum theory. Getting solutions of the equation of motion in a systematic way is important, especially in the case of non-Abelian gauge theory or gauge coupled Higgs theory. Because existence of the effect of the couplings even for the ground state is indispensable to understand the significant properties of the theory such as symmetry breaking or confinement. Although the classical solutions of the gauge theory were examined in various models, the topological solitons are particularly interesting from a point of view of the systematic construction of solutions.

The stability of such a solution is guaranteed by the topological properties of soliton. The field equations of non-Abelian gauge theory or of gauge coupled Higgs model are nonlinear second order differential equations, which are not integrable in general. However, there exist the first order equations, solutions of which automatically solve the second order field equations, and these solutions have the properties of topological soliton. Such a topological soliton equation is known as instanton equation for the Yang-Mills theory in R^4 or BPS(Bogomol'nyi-Prasad-Sommerfield) equation in the case of non-Abelian monopole in R^3 [1].

The instanton equation for the Yang-Mills theory in 4 dimensional Euclidean space R^4 is nothing but the self-duality equation for the field strength. The BPS equation in 3 dimensional space describes the static non-Abelian monopole of the Yang-Mills-Higgs theory, that is called BPS monopole, in the limit of vanishing Higgs coupling. The BPS monopole equation can be derived as a reduction of the self-dual Yang-Mills equation. Although other topological solitons in the gauge theory are also known, those have a lot of common properties. Generally, these solutions are called BPS solitons and the first order equations, to which they obey, are called BPS equations. One of the features of BPS equation which should be remarked is that we can minimize the Euclidean action or the energy integral by completing the square with these first order equations. The self-duality, although it changes its form in various cases, is inherent as the common property.

The ADHM (Atiyah-Drinfeld-Hitchin-Manin) construction of instantons is one of the most fruitful methods to obtain a soliton solution for the gauge theory [2]. This method translates the instanton moduli space, which is an information of the solutions of the self-dual Yang-Mills equation, into the space of the solutions of the algebraic equation (called hereafter ADHM equation). The BPS monopoles are obtained by the similar method which was proposed by Nahm [3]. In this case, the monopole moduli are determined by solving the first order ordinary differential equation (called Nahm equation).

In the recent decade, we have had a glimpse of new aspect of the ADHM/Nahm method. It was an interpretation as a configuration of D-branes. For example, the composite system of N D4-branes with k D0-branes can be seen as a configuration of k instantons for $\mathrm{U}(N)$ gauge theory in 4 dimensional space identified with the bundle of the D4-branes. In this case, the self-dual Yang-Mills equation and the ADHM equation are the conditions for supersymmetry in D4-brane and D0-brane respectively. In the case of monopoles, the system of N D3-branes with k D1-branes is interpreted as a configuration of k monopoles for $\mathrm{U}(N)$ gauge theory in 3 dimensions. The BPS equation and the Nahm equation are the SUSY conditions in D3-brane and D1-brane respectively. Here, in place of self-duality, a central role is played by supersymmetry.

During the last half of this decade, there appeared a new family of solitons, that is a non-Abelian vortex, adapted to the D-brane construction method of moduli [4]. The D-brane interpretation for the vortices can be given by the configuration that consists of N D3-branes suspended between two parallel NS5-branes. As a result of study in this direction, it has been pointed out that there seems to be close relationship between the moduli space of vortices and the moduli space of instantons. The vortex moduli space, in fact, involves half the elements of the ADHM construction and obeys the relation similar to the ADHM condition. Although these results were surely provided by a viewpoint of the D-brane and its supersymmetry, we do not understand the vortex moduli and the relation to the ADHM from a viewpoint of the field theory. The ADHM method allows us to construct the solutions to the self-dual Yang-Mills equation in a systematic way. It is interesting to look for a "self-duality" in the case of the vortex described by the "half-ADHM"[5].

We cast some light on the notion of self-duality of the vortex to understand the relation with the instanton. While the instanton equation expressed a self-duality for the Hodge operator, the vortex equation seems to have no more relation with the self-duality than that of being a first order BPS equation. Actually, we understand that this equation does express a self-duality by assuming appropriate space structure. We show that the non-Abelian vortex is nothing but the instanton in $R^2 \times Z_2$ space from the viewpoint of noncommutative differential geometry and gauge theory in discrete space [6 – 9]. Such an idea has been once proposed by Teo-Ting in the case of abelian model [10]. Here we adapt this method to the case of non-Abelian vortex as an extension. Then we clarified the reasons why many similarities are found in the methods for constructing the moduli of instanton and vortex.

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The constituents of this article are as follows. In section 2, we summarize some properties of the vortex. In section 3, we explain differential geometry and gauge theory in discrete noncommutative space. In section 4, we show the fact that the non-Abelian vortices in R^2 can be considered as the instantons in $R^2 \times Z_2$. Section 5 is assigned to the discussions.

2. Non-Abelian vortex and self-dual BPS equation

The vortex is a static solution of the Yang-Mills-Higgs system with a translational symmetry in one direction [11]. The configuration of the multi vortices consist of the individual elements with an axial symmetry around itself. We can consider the vortex in the cross section that is perpendicular to its axis. For example, we look upon the vortex in $3+1$ dimensions as a model in $2+1$ dimensions. From this viewpoint, the static vortex can be seen as a soliton solution in the 2 dimensional Euclidean space.

Let us summarize some properties of the vortex solution for Abelian Higgs model [12–14]. The Lagrangian of Abelian Higgs model in $2+1$ dimensions is given by

$$\mathcal{L} = \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{D_\mu\phi}D^\mu\phi + \frac{\lambda}{2}(|\phi|^2 - c)^2 \,. \tag{2.1}$$

It is known that there are topologically stable static solutions in this model in the case of $\lambda = 1$. Such static solutions called vortices are the configurations which minimize the energy integral

$$E = \int_{R^2} d^2x \left(\frac{1}{2}\left|F_{12}\right|^2 + |D_1\phi|^2 + |D_2\phi|^2 + \frac{1}{2}(|\phi|^2 - c)^2 \right) , \tag{2.2}$$

by satisfying the BPS equations

$$\begin{aligned} iF_{12} \pm \left(|\phi|^2 - c\right) &= 0 \,, \\ (D_1 \pm iD_2)\,\phi &= 0 \ . \end{aligned} \tag{2.3}$$

The equations are often called vortex equations. We can also regard the energy integral of the model in $2+1$ dimensional space-time as an action of the Euclidean version of a theory in $1+1$ dimensions. In such a case, the solutions of the BPS equations which give a minimum of the Euclidean action are also called vortices in 2 dimensional space.

In order to obtain the finite energy, it is necessary for these solutions to satisfy the boundary conditions

$$\begin{aligned} &|\phi|^2 \to c, \;\; D\phi \to 0, \\ &F_{12} \to 0 \end{aligned} \tag{2.4}$$

at $|x| \to \infty$ the spacial infinity which is identified with a circle S^1. This means that only pure gauge configurations are allowed at the spacial infinity. Therefore, the global properties of the solutions of the vortex equation are classified by the first homotopy group

$$\pi_1\left(U\left(1\right)\right) = \mathbb{Z} \ , \tag{2.5}$$

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representing the topological mapping index for $S^1 \to U(1)$. The integers corresponding to the elements of this homotopy group are given by

$$\frac{i}{2\pi}\int dx_1 dx_2 F_{12} = 0, \pm 1, \pm 2, \cdots . \tag{2.6}$$

This is nothing but the first Chern character of $U(1)$ gauge field and is the topological charge of the solutions called vortex number.

The model can be extended to the Yang-Mills-Higgs model which has non-Abelian gauge symmetry. Here we consider a $U(N_C)$ gauge group. Generally, we can also extend the model to have a flavor symmetry among N_F Higgs fields. In this case, the energy integral is of the form

$$E = \int dx_1 dx_2 \mathrm{Tr}\left(\frac{1}{2}|F_{12}|^2 + |D_1 H|^2 + |D_2 H|^2 + \frac{1}{2}\left(HH^\dagger - c\mathbf{1}_{N_C}\right)^2\right) \tag{2.7}$$

provided by the Lagrangian in $2+1$ dimensions

$$\mathcal{L} = \mathrm{Tr}\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D_\mu H)^\dagger D^\mu H + \frac{1}{2}(HH^\dagger - c\mathbf{1}_{N_C})^2\right). \tag{2.8}$$

Where, we define a covariant derivative D_μ and a field strength $F_{\mu\nu}$ as

$$D_\mu = \partial_\mu + A_\mu,\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\ , \tag{2.9}$$

and Tr is a trace over the adjoint representation of $U(N_C)$. It might be remarked that the gauge field A_μ and the field strength $F_{\mu\nu}$ are $N_C \times N_C$ anti-hermitian matrices and also that the Higgs field H is represented by a $N_C \times N_F$ matrix which means an array of N_F fundamental Higgs of $U(N_C)$.

The energy can be transformed into the form

$$E = \int dx_1 dx_2 \mathrm{Tr}\left(\frac{1}{2}\left|iF_{12} \pm \left(HH^\dagger - c\mathbf{1}_{N_C}\right)\right|^2 + |(D_1 \pm iD_2)H|^2\right) \pm i\int dx_1 dx_2 \mathrm{Tr}F_{12}\,, \tag{2.10}$$

omitting a surface integral which has no affect on account of the boundary conditions. The BPS equations minimizing the energy are

$$\begin{aligned} iF_{12} \pm \left(HH^\dagger - c\mathbf{1}_{N_C}\right) &= 0\,, \\ (D_1 \pm iD_2)H &= 0\,, \end{aligned} \tag{2.11}$$

in this case. These equations also have topologically stable solutions in a similar way as in the Abelian case, which we call non-Abelian vortices [4, 5]. It is also obvious that pure gauge configurations are allowed at the spacial infinity $|x| \to \infty$. It means that the topological property of the non-Abelian vortices is classified by the mapping index for $S^1 \to U(N_C)$. On account of the fact that $U(N_C)$ is equal to $U(1) \times SU(N_C)$, the corresponding homotopy group is

$$\pi_1(U(N_C)) = \pi_1(U(1)) = \mathbb{Z} \tag{2.12}$$

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whose elements are integers and are identified as vortex numbers given by

$$\frac{i}{2\pi}\int dx_1 dx_2 \mathrm{Tr} F_{12} = 0, \pm 1, \pm 2, \cdots \ . \tag{2.13}$$

Although this model has the local $U(N_C)$ gauge symmetry and the global $SU(N_F)$ flavor symmetry, there occurs the symmetry breaking due to the existence of the Higgs potential. The vacuum of the theory has completely broken symmetries and there appear vortex solutions, provided that $N_F \geq N_C$. In the case of $N_F = N_C$, these solutions are called local vortices and are expressed in terms of the moduli corresponding to positions besides internal symmetries. On the other hand, the solutions in case of $N_F > N_C$ are called semilocal vortices and require the moduli corresponding not only to the position but also to the size and orientation.

Provided that $z \equiv x_1 + i x_2$ is a complex coordinate for the 2 dimensional space R^2, then the solutions of the BPS equations are determined in general by the $N_C \times N_F$ matrix $H_0(z)$ which has elements consisting of holomorphic functions of z [5]. Here, H_0 is usually called a moduli matrix for the vortices and we can represent any solution of BPS equations by means of H_0 as follows. Let us introduce a $N_C \times N_C$ invertible matrix $S(z,\bar{z}) \in GL(N_C, \mathbb{C})$ and consider a gauge invariant quantity defined by $\Omega(z,\bar{z}) \equiv S(z,\bar{z}) S^\dagger(z,\bar{z})$. Then the Higgs and gauge fields should be written as

$$\begin{aligned} H &= S^{-1} H_0 \, , \\ A_1 + i A_2 &= 2 S^{-1} \bar{\partial}_z S \ . \end{aligned} \tag{2.14}$$

Actually, the first set of BPS equations could be solved for arbitrary S on account of these relations. And the second set of the BPS equations is written in the form of

$$\partial_z \left(\Omega^{-1} \bar{\partial}_z \Omega \right) = \frac{1}{2} \left(\Omega^{-1} H_0 H_0^\dagger - c \mathbf{1}_{N_C} \right) \ . \tag{2.15}$$

This equation is called *master equation* for the vortices and has a unique solution Ω for any given H_0. Here, S is determined except for the gauge degrees of freedom and some ambiguities of decomposition. As a result, we can find H and A_i which solve the BPS equation on account of the relations given above.

Let us consider the case of local vortex with $N_F = N_C \equiv N$. The moduli matrix $H_0(z)$ becomes a $N \times N$ matrix and it can be shown that the vortex number is given by [5]

$$k = \frac{1}{2\pi} \mathrm{Im} \oint dz \partial_z \log\left(\det H_0\right) \ . \tag{2.16}$$

This representation for the topological charge makes it clear that H_0 behaves like $\det H_0 \sim z^k$ at the spacial infinity $|x| \to \infty$. This agrees with the fact that the dimensions of the vortex moduli space is equal to $2kN$ known from the index theorem. It is known that the moduli space is constructed by the method which is called Kähler quotient, and is represented as

$$\{\boldsymbol{Z}, \boldsymbol{\Psi}\} // GL(k, \mathbb{C}) \simeq \left\{ (Z, \psi) \,|\, \left[Z^\dagger, Z \right] + \psi^\dagger \psi \propto \mathbf{1}_k \right\} / U(k) \, , \tag{2.17}$$

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where Z and ψ are $k \times k$ and $N \times k$ matrices respectively [4, 5]. This method to construct the vortex moduli extremely resembles the ADHM method to construct the instanton moduli and it is called half ADHM. It has not been clear why the moduli spaces of the vortex and the instanton are constructed by such similar methods. In the following sections, we will show that the non-Abelian vortex is equivalent to the instanton in $R^2 \times Z_2$.

3. Gauge theory in noncommutative discrete space

In order to make transparent the construction of the theory, we shall survey the method of representing the gauge theory in the noncommutative space in term of differential forms for the matrices. We propose that the gauge field is an extended differential $(p+q)$-form on $M \times Z_2$ space consisting of the (p,q)-forms on M and Z_2 respectively.

Let us consider discrete two point space Z_2 which has noncommutative nature in differential calculus [6 − 10]. We employ matrices as machineries representing such a structure and consider 2×2 matrices as differential forms in Z_2 space. Then we introduce Z_2-grading corresponding to the parity with respect to the degrees of differential forms. Where, the matrices with diagonal elements have even parity and the matrices with anti-diagonal elements have odd parity.

In general, 2×2 matrix

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{3.1}$$

should be interpreted as a mixed differential form consisting of different degrees of forms, which could be decomposed as $a = a_e + a_o$,

$$a_e = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}, \quad a_o = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} \ . \tag{3.2}$$

We represent the Z_2-parities of forms as $[a_e] = 0$ and $[a_o] = 1$ for even and odd matrices respectively. The wedge product among the differential forms is to be assumed as that of the matrices.

The exterior derivative operator d acting on the differential forms in Z_2 space is defined as

$$d = i\,[\eta, \quad \} \ , \tag{3.3}$$

where $[\alpha, \beta\}$ is the graded commutator representing

$$[\alpha, \beta\} = \alpha\beta - (-1)^{[\alpha][\beta]}\,\beta\alpha \tag{3.4}$$

and η is an odd parity matrix with the property

$$\eta^2 = \mathbf{1}\ . \tag{3.5}$$

Then, the action of d on arbitrary matrix differential form α is

$$d\alpha = i\,[\eta, \alpha\} = i\left(\eta\alpha - (-1)^{[\alpha]}\,\alpha\eta\right) \ . \tag{3.6}$$

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This leads to the result

$$d^2\alpha = -[\eta, [\eta, \alpha\}\} = \frac{1}{2}[\alpha, [\eta, \eta\}\} = 0\,, \tag{3.7}$$

where we use the relation

$$2[\eta, [\eta, \alpha\}\} + [\alpha, [\eta, \eta\}\} = 0 \tag{3.8}$$

derived from the graded Jacobi identity

$$(-1)^{[A][C]}[A, [B, C\}\} + (-1)^{[A][B]}[B, [C, A\}\} + (-1)^{[C][B]}[C, [A, B\}\} = 0 \tag{3.9}$$

and the fact that $\eta^2 = \mathbf{1}$. Then we can consider d itself as nilpotent

$$d^2 = 0\,, \tag{3.10}$$

which means that d plays a role of an exterior derivative operator as it is a linear operator with odd parity and has the nilpotency. The graded Leipniz's rule

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{[\alpha]}\alpha \wedge d\beta \tag{3.11}$$

is one of the desirable properties.

Although we may employ the form of

$$\eta = \eta_\gamma = \cos\gamma\tau_1 + \sin\gamma\tau_2 = \begin{pmatrix} 0 & e^{-i\gamma} \\ e^{i\gamma} & 0 \end{pmatrix}, \tag{3.12}$$

as η in general, it is convenient to adopt

$$\eta = \eta_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \tau_1 \tag{3.13}$$

to make calculations clear without loss of generality. We shall follow this definition in this paper for simplicity, then we see that the action of d on the matrix a results

$$da = i[\eta, a_e] + i\{\eta, a_o\} = i\begin{pmatrix} a_{21} + a_{12} & a_{22} - a_{11} \\ a_{11} - a_{22} & a_{21} + a_{12} \end{pmatrix}. \tag{3.14}$$

Let us consider the differential forms in an extended space $M \times Z_2$ with M an ordinary manifold. These are represented by 2×2 matrices whose elements are consisting of differential forms in M. For example, if we consider the extended differential forms

$$\mathcal{M} = \begin{pmatrix} A & C \\ D & B \end{pmatrix}, \ \mathcal{M}' = \begin{pmatrix} A' & C' \\ D' & B' \end{pmatrix}, \tag{3.15}$$

then the wedge product of these is a product of two matrices accounting the signature of grading as follows

$$\mathcal{M} \wedge \mathcal{M}' = \begin{pmatrix} A \wedge A' + (-1)^{[C]}C \wedge D' & (-1)^{[A]}A \wedge C' + C \wedge B' \\ D \wedge A' + (-1)^{[B]}B \wedge D' & (-1)^{[D]}D \wedge C' + B \wedge B' \end{pmatrix}. \tag{3.16}$$

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Here, $[A]$ stands for the Z_2-parity of degree of the differential form A. The rule for the wedge product given above means that Z_2-parity of the differential forms in M should be identified with that of the 2×2 matrices in Z_2.

The arbitrary matrix $\mathcal{M}$ is represented as

$$\mathcal{M} = e_{ij} \otimes \mathcal{M}_{ij} \,, \tag{3.17}$$

provided that the basis of 2×2 matrices are assigned as

$$e_{00} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; e_{01} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; e_{10} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; e_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.18}$$

with the Z_2-parity $[e_{00}] = [e_{11}] = 0$ (even), $[e_{01}] = [e_{10}] = 1$ (odd). Here we employ a representation in terms of the direct product. And keep a convention that the basis of 2×2 matrix forms should be located on the left of the ordinary differential forms. Then the wedge product of the two matrix differential forms $\mathcal{M}$ and $\mathcal{M}'$ is calculated as

$$\begin{aligned} \mathcal{M} \wedge \mathcal{M}' &= (e_{ij} \otimes \mathcal{M}_{ij}) \wedge \left(e_{kl} \otimes \mathcal{M}'_{kl}\right) \\ &= (e_{ij}\; e_{kl}) \otimes \left((-1)^{[\mathcal{M}_{ij}][e_{kl}]}\, \mathcal{M}_{ij} \wedge \mathcal{M}'_{kl}\right) \\ &= (\delta_{jk}\; e_{il}) \otimes \left((-1)^{[\mathcal{M}_{ij}][e_{kl}]}\, \mathcal{M}_{ij} \wedge \mathcal{M}'_{kl}\right) \\ &= e_{il} \otimes \left((-1)^{[\mathcal{M}_{ij}][e_{jl}]}\, \mathcal{M}_{ij} \wedge \mathcal{M}'_{jl}\right) \;. \end{aligned} \tag{3.19}$$

It is worth to be remarked that the same sign rule

$$\alpha \wedge \beta = (-1)^{[\alpha][\beta]}\, \beta \wedge \alpha \tag{3.20}$$

should be applied when we exchange the order between the basis of matrix differential forms and the ordinary differential forms. There exists similar idea to define the hermitian conjugate of the graded matrix differential form. It is defined by

$$\mathcal{M}^\dagger = \begin{pmatrix} A^\dagger & (-1)^{[D]}\, D^\dagger \\ (-1)^{[C]}\, C^\dagger & B^\dagger \end{pmatrix} \tag{3.21}$$

for $\mathcal{M}$ given above.

To make the difference clear, let us use the symbols d_H and d_V to represent the exterior derivative operators on M (horizontal) and Z_2 (vertical) spaces respectively. That is to say, d_H represents the exterior derivative operator acting on the ordinary differential forms and d_V represents the exterior derivative on the matrix differential forms. If we identify the Z_2-parity of the matrix differential forms with that of the ordinary differential forms, then we can agree that the operators d_H and d_V anti-commute with each other

$$d_H d_V = -d_V d_H \;. \tag{3.22}$$

Thus, we consider the exterior derivative operator acting on the generalized differential form in $M \times Z_2$

$$\boldsymbol{d} = d_H + d_V \tag{3.23}$$

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which satisfies the nilpotency as

$$\boldsymbol{d}^2 = d_H^2 + d_H d_V + d_V d_H + d_V^2 = 0\ . \tag{3.24}$$

The explicit action of $\boldsymbol{d}$ on the matrix differential form $\mathcal{M}$ is

$$\begin{aligned}\boldsymbol{d}\mathcal{M} &= d_H\mathcal{M} + d_V\mathcal{M}\\ &= \begin{pmatrix} dA & -dC \\ -dD & dB \end{pmatrix} + i\left[\eta, \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right] + i\left\{\eta, \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}\right\}\\ &= \begin{pmatrix} dA + i\,(C+D) & -dC - i\,(A-B) \\ -dD + i\,(A-B) & dB + i\,(C+D) \end{pmatrix},\end{aligned} \tag{3.25}$$

where we simply write d in place of d_H when its role is manifest.

Actually, we can assign not only the Z_2-parity but also the degree to matrix differential forms in Z_2 space. In general, 2×2 matrix can be expanded in terms of the basis $(\mathbf{1}, \tau_1, \tau_2, \tau_3)$ with Pauli matrices τ_i 's. These bases are classified into the even basis $(1, \tau_3)$ and odd one (τ_1, τ_2) . It is natural to consider $\mathbf{1}$ as a basis of 0-form and (τ_1, τ_2) as basis of 1-form according to their Z_2-parity. The basis of 1-form is often written as (θ_1, θ_2). Then the basis of 2-form can be obtained by

$$\theta_1 \wedge \theta_2 = \tau_1\tau_2 = i\tau_3\ . \tag{3.26}$$

The 2×2 matrix $\mathcal{M}$ is expanded as

$$\mathcal{M} = \begin{pmatrix} A & C \\ D & B \end{pmatrix} = \mathbf{1}\otimes\frac{A+B}{2} + \tau_1\otimes\frac{C+D}{2} + \tau_2\otimes\frac{i\,(C-D)}{2} + \tau_3\otimes\frac{A-B}{2}, \tag{3.27}$$

then we can interpret these terms as different degrees of forms in Z_2 space.

We can define a gauge theory in $M\times Z_2$ in terms of the matrix differential forms. Let us consider a gauge field as a connection 1-form in $M\times Z_2$ space in the form of

$$\mathcal{A} = \begin{pmatrix} L & i\varphi \\ i\varphi^\dagger & R \end{pmatrix}, \tag{3.28}$$

where L, R and φ are Lie-algebra valued 1-forms and 0-form on M respectively. It should be remarked that L and R are anti-hermitian and φ is complex. Let our model be that consisting of L, R, φ with value on $N\times N$ matrices. It means that the model has the $U\,(N)_L \times U\,(N)_R$ gauge symmetry. The connection form is expanded as

$$\mathcal{A} = \mathcal{A}^{(1,0)} + \mathcal{A}^{(1,2)} + \mathcal{A}^{(0,1)}, \tag{3.29}$$

where

$$\begin{aligned}\mathcal{A}^{(1,0)} &= \mathbf{1}\otimes\frac{L+R}{2},\\ \mathcal{A}^{(1,2)} &= \tau_3\otimes\frac{L-R}{2},\\ \mathcal{A}^{(0,1)} &= \tau_1\otimes\frac{\varphi+\varphi^\dagger}{2} + \tau_2\otimes\frac{i\,(\varphi-\varphi^\dagger)}{2}.\end{aligned} \tag{3.30}$$

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We use the notation $\mathcal{A}^{(p,q)}$ to represent a form that behaves itself as a p-form in M and as a matrix q-form in Z_2 , that is, a $(p+q)$-form in $M \times Z_2$ space as a whole.

The field strength $\mathcal{F}$ derived from the gauge field $\mathcal{A}$ is defined by

$$\mathcal{F} = \boldsymbol{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \tag{3.31}$$

or

$$\begin{aligned}
\mathcal{F} &= \begin{pmatrix} dL + L \wedge L - \varphi\varphi^\dagger - \left(\varphi + \varphi^\dagger\right) & -i\left(d\varphi + (L\varphi - \varphi R) + (L - R)\right) \\ -id\left(d\varphi^\dagger - \left(\varphi^\dagger L - R\varphi^\dagger\right) - (L - R)\right) & dR + R \wedge R - \varphi^\dagger\varphi - \left(\varphi + \varphi^\dagger\right) \end{pmatrix} \\
&= \begin{pmatrix} F^L - W^L & -iD\phi \\ -i\left(D\phi\right)^\dagger & F^R - W^R \end{pmatrix}
\end{aligned} \tag{3.32}$$

in components. Where, F^L and F^R are field strengths of the gauge fields L and R on the manifold M,

$$\begin{aligned}
F^L &= dL + L \wedge L\,, \\
F^R &= dR + R \wedge R\,.
\end{aligned} \tag{3.33}$$

D is a covariant derivative with respect to both L and R ,

$$D\phi = d\phi + L\phi - \phi R\,, \tag{3.34}$$

provided that L and R act from the left and right respectively. ϕ is defined by

$$\phi = \varphi + \mathbf{1}_N\,, \tag{3.35}$$

then W^L and W^R are defined by

$$\begin{aligned}
W^L(\phi) &= (\varphi + \mathbf{1}_N)\left(\varphi^\dagger + \mathbf{1}_N\right) - \mathbf{1}_N = \phi\phi^\dagger - \mathbf{1}_N\,, \\
W^R(\phi) &= \left(\varphi^\dagger + \mathbf{1}_N\right)(\varphi + \mathbf{1}_N) - \mathbf{1}_N = \phi^\dagger\phi - \mathbf{1}_N\,.
\end{aligned} \tag{3.36}$$

As we shall see in the following section, Higgs potential $V(\phi)$ can be given by W^L and W^R as

$$V(\phi) = \mathrm{Tr}\left(W^L\right)^2 = \mathrm{Tr}\left(W^R\right)^2 = \mathrm{Tr}\left(\phi\phi^\dagger - \mathbf{1}_N\right)^2 = \mathrm{Tr}\left(\phi^\dagger\phi - \mathbf{1}_N\right)^2\,, \tag{3.37}$$

where Tr is a trace over the representation matrix of Lie-algebra. It means that ϕ is the Higgs filed and that φ is its fluctuation around the vacuum expectation value $\mathbf{1}_N$.

There exists an appropriate definition of the Hodge dual ${}^*\mathcal{F}$ of $\mathcal{F}$ and a definition of volume integral of norm square of $\mathcal{F}$ on $M \times Z_2$ space, that is

$$\mathrm{Tr}\int_{M\times Z_2} \langle \mathcal{F}, \mathcal{F}\rangle = \mathrm{Tr}\int_{M\times Z_2} \mathcal{F} \wedge^* \mathcal{F}\,. \tag{3.38}$$

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The action of the gauge theory is nothing but this integral and we have

$$\begin{aligned}S=&\frac{1}{2}\mathrm{Tr}\int_{M\times Z_2}\mathcal{F}\wedge^{*}\mathcal{F}\\=&\mathrm{Tr}\int_M\left(\frac{1}{2}\left|F_{12}^L\right|^2+\frac{1}{2}\left|F_{12}^R\right|^2+\left(\phi\phi^\dagger-\mathbf{1}_N\right)^2+\left|D_1\phi\right|^2+\left|D_2\phi\right|^2\right)dx^1\wedge dx^2\wedge\cdots\wedge dx^n\end{aligned} \tag{3.39}$$

as a result. The concrete definition of Hodge duality is necessary to express the Yang-Mills action by means of the inner product among the differential forms on the noncommutative space. The derivation of the above action with the definite form of Hodge duality will be explained in some detail in the next section.

This action is that of the Yang-Mills-Higgs model, which consists of the kinetic terms of the Yang-Mills gauge field and the Higgs field. This means that pure Yang-Mills gauge theory in $M\times Z_2$ is equivalent to the Yang-Mills-Higgs theory in M which is automatically incorporated in the mechanism of spontaneous symmetry breaking in natural way.

Let us think of the case of $N=1$ that is abelian gauge theory for simplicity, we see that the combination of $L+R$ becomes massive and $L-R$ remains massless. That is to say, this is a model of the Higgs mechanism which breaks a gauge symmetry $U(1)_L\times U(1)_R$ to $U(1)_{L-R}$. If we adapt this machinery to the standard model, it would be suitable to assign $U(2)_L\times U(1)_R$ as a gauge group, with total trace free condition.

In this construction, we can understand that the Higgs field is included as a kind of gauge field by generalizing the gauge theory to the space with discrete and noncommutative geometry. Then this method leads to the Higgs mechanism and spontaneous symmetry breaking naturally, which is nothing but the gauge theory itself. There has been many explicit applications of this idea to reconstruct standard model [6–8].

4. Non-abelian vortex on R^2 as instanton on $R^2\times Z_2$

The idea of Hodge duality plays a crucial role in understanding the relation of instantons and vortices. Actually, to interpret the non-Abelian vortex on R^2 as an instanton on $R^2\times Z_2$, the concept of Hodge duality for the matrix differential forms on the noncommutative space has to be defined. This concept, which seems to be not necessarily well defined in the literature, is indispensable.

In this section, we have worked out the concrete definition of Hodge duality. Based on this, we have expressed the Yang-Mills action, which we have explained in the previous section, on the noncommutative space in terms of the Hodge dual operation. Furthermore under the operation of the Hodge dual, we describe the instanton equation on $R^2\times Z_2$, and reveal the fact that it is equivalent to the vortex equation on R^2.

The general p-form in the ordinary manifold M can be written as

$$\alpha=\frac{1}{p!}\alpha_{\mu_1\mu_2\cdots\mu_p}dx^{\mu_1}\wedge dx^{\mu_2}\wedge\cdots\wedge dx^{\mu_p} \tag{4.1}$$

in terms of the basis dx^μ of the covariant vectors which span the cotangent vector bundle $T^*(M)$. One can also introduce the dual basis $\dfrac{\partial}{\partial x^\mu}$ to dx^μ , that is, the basis of the

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contravariant vectors which span the tangent vector bundle $T(M)$ of the manifold M. The inner product among the basis and dual one is defined to satisfy the relation

$$\left\langle dx^\mu, \frac{\partial}{\partial x^\nu} \right\rangle = \delta^\mu_\nu \ . \tag{4.2}$$

The Hodge dual operation $*$ is defined so as to transfer the inner product of α and β to the wedge product of α and ${}^*\beta$

$$\langle \alpha, \beta \rangle \, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n = \alpha \wedge {}^*\beta \ . \tag{4.3}$$

Let us define the explicit correspondence in terms of the Hodge dual of the forms in R^2 as follows. The Hodge dual of the 2-form F, 1-form V and 0-form W,

$$\begin{aligned} F &= \frac{1}{2} F_{ij} dx^i \wedge dx^j \ , \\ V &= V_i dx^i \ , \end{aligned} \tag{4.4}$$

are given by the 0-form ${}^*F$, 1-form ${}^*V$ and 2-form ${}^*W$,

$$\begin{aligned} {}^*F &= \frac{1}{2} \varepsilon_{ji} F_{ij} = -F_{12} \ , \\ {}^*V &= \varepsilon_{ij} V_j dx^i \ , \\ {}^*W &= \frac{1}{2} \varepsilon_{ij} W dx^i \wedge dx^j \ , \end{aligned} \tag{4.5}$$

respectively. As a result of the definition given above, we see that ${}^{**} = -1$ for the forms of arbitrary degrees. This means that our definition results in ${}^{**} = 1$ when it is extended to the case of 4 dimensional space R^4 with Euclidean signature.

As the eigen values of the Hodge duality operator $*$ in R^4 are ± 1, we can define the (anti-)selfdual 2-form F_+ (F_-) by

$$F_\pm = F \pm {}^*F \tag{4.6}$$

which are the eigen states of the operator $*$

$${}^*F_\pm = \pm F_\pm \tag{4.7}$$

with respect to an arbitrary 2-form F in R^4 .

On the other hand, the eigen values of the Hodge duality operator $*$ in R^2 are $\pm i$. Then we can define the (anti-)selfdual 1-form V_+ (V_-) by

$$V_\pm = V \mp i{}^*V \tag{4.8}$$

which are the eigen states of the operator $*$

$${}^*V_\pm = \pm i V_\pm \tag{4.9}$$

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with respect to an arbitrary 1-form V in R^2. $V_\pm$ has the components $V_\pm = ((V_\pm)_1, (V_\pm)_2)$

$$(V_\pm)_1 = V_1 \mp iV_2\,,$$
$$(V_\pm)_2 = V_2 \pm iV_1 = \pm i\,(V_1 \mp iV_2) = \pm i\,(V_\pm)_1\,, \tag{4.10}$$

with $V = (V_1, V_2)$.

Let us consider the case of the matrix differential forms. As the basis θ^a of the covariant vectors are represented by the matrices, the dual basis or the contravariant vectors e_a which satisfy the relation

$$\langle \theta^a, e_b \rangle = \delta^a_b\,, \tag{4.11}$$

are also represented by matrices. We can see that this inner product is the normalized trace of such matrices. Actually in case of matrix differential forms in Z_2, as we employ the convention that the basis are represented by $\theta^1 = \tau_1,\ \theta^2 = \tau_2$, the dual basis have the same forms as themselves,

$$e_1 = \tau_1,\ e_2 = \tau_2. \tag{4.12}$$

In this representation, the Hodge dual operation is equal to the multiplication by $i\tau_3$. This operation maps the basis of the matrix $0,1,2$-forms, $\left\{1, \left(\theta^1, \theta^2\right), \theta^1 \wedge \theta^2\right\}$ or $\{1, (\tau_1, \tau_2), i\tau_3\}$ to the $2,1,0$-forms, $\left\{\theta^1 \wedge \theta^2, \left(\theta^2, -\theta^1\right), -1\right\}$ or $\{i\tau_3, (\tau_2, -\tau_1), -1\}$. These results coincide with that of the operation $*$ in R^2 described above. Thus we can see that the Hodge dual in $R^2 \times Z_2$ as a four-dimensional space is performed by the duality operation in R^2 and Z_2 at the same time. As a result, we obtain real values ± 1 as the eigen values of the Hodge duality operator in $R^2 \times Z_2$, although those values are imaginary $\pm i$ in R^2 and Z_2.

Let us consider the pure Yang-Mills action in $R^2 \times Z_2$. The Hodge dual of field strength 2-form for the gauge field in $R^2 \times Z_2$ is given by

$${}^*\mathcal{F} \equiv i\tau_3 \mathcal{F}\,({}^*)\ . \tag{4.13}$$

Here, we mean $\mathcal{F}\,({}^*)$ the Hodge dual with respect to the forms in R^2 as the components of 2×2 matrix $\mathcal{F}$, whereas the Hodge dual with respect to the matrix differential forms in Z_2 is represented by the multiplication with $i\tau_3$.

We could obtain the action for pure Yang-Mills theory on $R^2 \times Z_2$ by integration of the Lagrangian

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}\,\langle \mathcal{F}, \mathcal{F} \rangle = \frac{1}{2}\mathrm{Tr}\mathcal{F} \wedge^* \mathcal{F} \tag{4.14}$$

over the volume of $R^2 \times Z_2$. Although different degrees of forms coexist in $\mathcal{F} \wedge^* \mathcal{F}$, we would pick up the volume form $dx^1 \wedge dx^2 \wedge \theta^1 \wedge \theta^2$ out of it. Accounting the fact that the volume form $\theta^1 \wedge \theta^2$ of the Z_2 space is equal to $i\tau_3$ in our convention, we have to take a trace of $\mathcal{L}$ after multiplying by $-i\tau_3$ as a 2×2 matrix, in order to pick up the volume form. Thus the volume integral $\int_{Z_2}(\ )$ over the Z_2 space is equivalent to $-\frac{i}{2}\ \mathrm{Tr}_{Z_2}\tau_3\,(\ )$, where Tr_{Z_2} represents a trace with respect to 2×2 matrix as a differential form in Z_2 space. It would be clear that the volume integral $\int_{R^2}(\ )$ over the R^2 leaves 2-forms. As a results, we have the action in the form of

$$
\begin{aligned}
S &= \mathrm{Tr}\int_{R^2\times Z_2}\mathcal{F}\wedge^*\mathcal{F}\\
&= \frac{1}{2}\mathrm{Tr}\int_{R^2}\mathrm{Tr}_{Z_2}\tau_3\mathcal{F}\wedge\tau_3\mathcal{F}\,(^*)\\
&= \frac{1}{2}\mathrm{Tr}\int_{R^2}\mathrm{Tr}_{Z_2}\tau_3\begin{pmatrix} F^L-W^L & -iD\phi\\ -i\,(D\phi)^\dagger & F^R-W^R\end{pmatrix}\wedge\begin{pmatrix} ^*\left(F^L-W^L\right) & -i^*\,(D\phi)\\ i^*\,(D\phi)^\dagger & -^*\left(F^R-W^R\right)\end{pmatrix}\\
&=\mathrm{Tr}\int_{R^2}\left(\frac{1}{2}\left|F^L_{12}\right|^2+\frac{1}{2}\left|F^R_{12}\right|^2+\left(\phi\phi^\dagger-\mathbf{1}_N\right)^2+|D_1\phi|^2+|D_2\phi|^2\right)dx^1\wedge dx^2. \qquad (4.15)
\end{aligned}
$$

Thus we can confirm the fact that the action for pure Yang-Mills theory in $R^2\times Z_2$ is equivalent to the action for Yang-Mills-Higgs theory in R^2.

We consider a model with $U(N)_L\times U(N)_R$ gauge symmetry, where the fields L, R and φ are $N\times N$ matrices in general. In order to obtain a model for the non-Abelian vortex considered in ref's [4, 5], it would be required to make an appropriate reduction, that is to restrict the gauge group to $U(N)_L\times U(1)_R$. Then we combine $U(1)_R$ with $U(1)_L$, that is a subgroup of $U(N)_L$, to obtain $U(1)_{L-R}$ and $U(1)_{L+R}$, the latter of which is decoupled from the other fields. If we discard the decoupled $U(1)$, we have a model with $U(N)$ gauge symmetry, which describes local vortex. Our general model is considered as that with the extension to have a local flavor symmetry, which should be frozen to become a global one.

Now, we shall show that the instanton on $R^2\times Z_2$ is equivalent to the vortex on R^2. The field strength "2-form" $\mathcal{F}$ of the gauge field on $\mathcal{M}_4 = R^2\times Z_2$ can be decomposed as

$$
\mathcal{F}=\mathcal{F}^{(0,0)}+\mathcal{F}^{(2,0)}+\mathcal{F}^{(1,1)}+\mathcal{F}^{(0,2)}+\mathcal{F}^{(2,2)}, \qquad (4.16)
$$

provided that the basis of $0,1,2$-forms on Z_2 are $\mathbf{1},(\tau_1,\tau_2),i\tau_3$ respectively, where

$$
\begin{aligned}
\mathcal{F}^{(0,0)} &= \mathbf{1}\otimes\left(-\frac{W^L+W^R}{2}\right),\\
\mathcal{F}^{(2,0)} &= \mathbf{1}\otimes\left(\frac{F^L+F^R}{2}\right),\\
\mathcal{F}^{(1,1)} &= -i\left(\tau_1\otimes\left(\frac{D\phi+(D\phi)^\dagger}{2}\right)+\tau_2\otimes i\left(\frac{D\phi-(D\phi)^\dagger}{2}\right)\right),\\
\mathcal{F}^{(0,2)} &= i\tau_3\otimes(-i)\left(-\frac{W^L-W^R}{2}\right),\\
\mathcal{F}^{(2,2)} &= i\tau_3\otimes(-i)\left(\frac{F^L-F^R}{2}\right). \qquad (4.17)
\end{aligned}
$$

With respect to the total degrees, $\mathcal{F}$ should be understood as an mixed form consisting of not only total 2-form $\mathcal{F}^{(2,0)}+\mathcal{F}^{(1,1)}+\mathcal{F}^{(0,2)}$ but also 0-form $\mathcal{F}^{(0,0)}$ and 4-form $\mathcal{F}^{(2,2)}$.

The Hodge operator $*$ in $R^2\times Z_2$ transfers the field strength $\mathcal{F}$ into its dual $^*\mathcal{F}\equiv i\tau_3\mathcal{F}\,(^*)$ according to the definition in the previous section. As a result, the components of $\mathcal{F}$ in the above decomposition are transferred as

$$
\mathcal{F}^{(p,q)}\to\ ^*\mathcal{F}^{(2-p,2-q)}. \qquad (4.18)
$$

Then we can see that there is a correspondence not only between 2-form and dual 2-form but also between 0-form and 4-form. The instanton equation for Yang-Mills gauge field in 4 dimensions is nothing but a requirement of (anti-)selfduality for the field strength 2-form,

$$ {}^{*}\mathcal{F} = \pm \mathcal{F}. \tag{4.19} $$

For the case of the gauge field in $R^2 \times Z_2$, the (anti-)selfdual Yang-Mills equation means the correspondence of the form

$$ {}^{*}\mathcal{F}^{(p,q)} = \pm \mathcal{F}^{(2-p,2-q)}, \tag{4.20} $$

that is,

$$ i\tau_3 \mathcal{F}^{(p,q)}\,({}^{*}) = \pm \mathcal{F}^{(2-p,2-q)}, \tag{4.21} $$

on account of nature of the Hodge operator for the matrix differential forms. This equation is decomposed as follows in terms of the differential forms in R^2

$$ {}^{*}\left(\frac{F^L + F^R}{2}\right) = \pm(-i)\left(-\frac{W^L - W^R}{2}\right) \tag{4.22} $$

for $(p,q) = (2,0)$ or $(0,2)$,

$$ {}^{*}\left(-\frac{W^L + W^R}{2}\right) = \pm(-i)\left(\frac{F^L - F^R}{2}\right) \tag{4.23} $$

for $(p,q) = (0,0)$ or $(2,2)$, and

$$ \begin{aligned} (i\tau_3)&\left(\tau_1 \otimes {}^{*}\left(\frac{D\phi + (D\phi)^\dagger}{2}\right) + \tau_2 \otimes i\, {}^{*}\left(\frac{D\phi - (D\phi)^\dagger}{2}\right)\right) \\ &= \pm\left(\tau_1 \otimes \left(\frac{D\phi + (D\phi)^\dagger}{2}\right) + \tau_2 \otimes i\left(\frac{D\phi - (D\phi)^\dagger}{2}\right)\right) \end{aligned} \tag{4.24} $$

for $(p,q) = (1,1)$. As a result, we have the (anti-)selfdual equations written as

$$ \begin{aligned} iF^L_{12} \pm W^L &= 0, \\ iF^R_{12} \mp W^R &= 0, \\ (D_1 \pm iD_2)\,\phi &= 0, \end{aligned} \tag{4.25} $$

in components.

These equations can also be obtained by considering on the equal footing the differential forms of different nature. Suppose the basis of 1-form in "4 dimensional" space $\mathcal{M}_4 = R^2 \times Z_2$ to be dx^μ ($\mu = 1,2,3,4$). Let us consider indices 1,2 to show ingredients of basis of the 1-form in 2 dimensional continuous space R^2 and index 3,4 to show those in "2 dimensional" discrete space Z_2. That means assigning

$$ \left(dx^3, dx^4\right) = \left(\tau^1, \tau^2\right) \tag{4.26} $$

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besides the ordinary basis $\left(dx^1, dx^2\right)$. On account of these assignments, we can also obtain the (anti-)selfdual Yang-Mills equations in the same way as the usual expression in terms of the components.

This is a BPS equation expressing the non-Abelian vortex, so we have shown that an instanton equation in $R^2 \times Z_2$ was none other than the non-Abelian vortex equation in R^2. It can also be verified that the instanton number or a 2nd Chern character in $R^2 \times Z_2$ is just the vortex number in R^2 as follows. Remember the volume integral over the Z_2 is equal to the trace after multiplication with $-\frac{i}{2}\tau_3$, then we have the relation

$$-\mathrm{Tr}\int_{R^2\times Z_2} \mathcal{F}\wedge\mathcal{F} = i\int_{R^2}\frac{1}{2}\mathrm{Tr}_{Z_2}\tau_3\mathcal{F}\wedge\mathcal{F} = i\int_{R^2}\left(\mathrm{Tr}F^L_{12} - \mathrm{Tr}F^R_{12}\right)dx^1\wedge dx^2, \tag{4.27}$$

which means that instanton number on $R^2 \times Z_2$ is given by the difference between the vortex numbers of two gauge fields on R^2. Actually $\mathrm{Tr}F^L_{12}$ and $\mathrm{Tr}F^R_{12}$ have opposite sign, when we consider non trivial vortex solutions. As a result, we have vortex number as an instanton number in noncommutative discrete space.

5. Discussion

In this article, we have employed a matrix differential form to express differential geometry of noncommutative discrete space. As has been described in this work, the non-Abelian vortex in R^2 is equivalent to the instanton on the $R^2 \times Z_2$. This suggests the possibility to constitute non-Abelian vortex solution by the ADHM method. Actually, the moduli for the non-Abelian vortices are described by the method that resembles ADHM which is named "half ADHM", though the explicit form of the solutions is not decided yet. The relation between the instanton and the vortex that we reported in this article shows possibility to clarify the reason why the half-ADHM method works.

In the usual ADHM method for instanton [15], we employ a quaternionic variables x as a coordinate of 4 dimensional space R^4. The ADHM data, that is, moduli parameters to describe instantons, are included into the "0 dimensional Dirac operator" $\nabla = Cx - D$ as its coefficient matrices, C and D. We should solve the "Dirac equation" $\nabla^\dagger V = 0$, in order to determine the gauge connection in the form of $A = V^\dagger dV$ with V which satisfies $V^\dagger V = 1$. The condition for the field strength to be anti-selfdual is that $\nabla^\dagger\nabla$ should be an invertible matrix which consists of the real numbers although ∇ itself has quaternionic entities.

Although differential forms and the calculation rule among them are given in the case of $R^2 \times Z_2$ space, we do not know an exact expression for the coordinates in this space. The operator ∇ therefore is not yet given which is a key issue to clarify how to construct the ADHM in this case. It is suggested that there exists a mechanism analogous to ADHM even if the exact expression of the coordinate is unidentified. For example, the gauge field is given in the form of a kind of non-linear sigma models. If we tentatively assume that the extension of the V in R^2 to $R^2 \times Z_2$ is given by

$$\mathcal{V} = \begin{pmatrix} V_L & 0 \\ 0 & V_R \end{pmatrix},\; V_L^\dagger V_L = 1,\; V_R^\dagger V_R = 1 \tag{5.1}$$

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with appropriate matrices V_L and V_R. Then the gauge connection becomes

$$\mathcal{A} = \mathcal{V}^\dagger d\mathcal{V} = \begin{pmatrix} V_L^\dagger dV_L & i\left(V_L^\dagger V_R - 1\right) \\ i\left(V_R^\dagger V_L - 1\right) & V_R^\dagger dV_R \end{pmatrix}. \tag{5.2}$$

We can see that the gauge connection discussed in this article is obtained with the assignment

$$\begin{aligned} &L = V_L^\dagger dV_L,\ R = V_R^\dagger dV_R, \\ &\varphi = V_L^\dagger V_R - 1, \\ &\phi = V_L^\dagger V_R. \end{aligned} \tag{5.3}$$

Therefore, we can complete the ADHM method, if we can introduce suitable coordinate expression.

It is also natural to expect that the ADHM is applicable here on the ground that there has been an analogy between the Yang equation for instanton and the master equation for vortex. Let $z \equiv x_1 + ix_2$, $w \equiv x_3 + ix_4$ be complex coordinates in R^4, then the instanton equation, that is, the anti-selfdual Yang-Mills equation is equivalent to the Yang equation

$$\partial_z\left(J^{-1}\bar{\partial}_z J\right) + \partial_w\left(J^{-1}\bar{\partial}_w J\right) = 0 \tag{5.4}$$

for the Yang's potential J [16]. It is obvious that the master equation is the analog of the Yang equation. The relations would be explained, if we can regard w as a coordinate of the Z_2 space and could assign appropriate expression to them.

Instanton equation in the usual space can be completely solved by the ADHM method. As for the vortex equation on the other hand, although it can be rewritten as a master equation plus half-ADHM, the solution cannot be obtained because we are left with the master equation. We consider however, that the difference can be attributed to the structure of Z_2 space. And in order to understand the situation, we have to have the representation of not only the differential forms but the representation of the coordinates. For the differential forms, we are using the matrix representation, but the representation for the background noncommutative coordinates should be considered separately. A construction method in terms of the coordinate will appear in the work in preparation.

In this article, we have shown that the non-Abelian vortex in R^2 is equivalent to the instanton on $R^2 \times Z_2$ space. It has been proposed in ref [17], that there exists similar relation in the case of the model on compact Riemann surface Σ. They have shown that the instanton on $\Sigma \times CP^1$ can be considered as a non-Abelian vortex on Σ. It would be interesting to examine the relations between our work and their approach.

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References

[1] E.B. Bogomol'ny, *Stability of classical solutions*, *Sov. J. Nucl. Phys.* **24** (1976) 449 [*Yad. Fiz.* **24** (1976) 861];
M.K. Prasad and C.M. Sommerfield, *An exact classical solution for the 't Hooft monopole and the Julia-Zee dyon*, *Phys. Rev. Lett.* **35** (1975) 760.

[2] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Y.I. Manin, *Construction of instantons*, *Phys. Lett.* **A 65** (1978) 185.

[3] W. Nahm, *A simple formalism for the BPS monopole*, *Phys. Lett.* **B 90** (1980) 413.

[4] D. Tong, *The moduli space of noncommutative vortices*, *J. Math. Phys.* **44** (2003) 3509 [hep-th/0210010];
A. Hanany and D. Tong, *Vortices, instantons and branes*, *JHEP* **07** (2003) 037 [hep-th/0306150];
D. Tong, *TASI lectures on solitons*, hep-th/0509216;
N. Sakai and D. Tong, *Monopoles, vortices, domain walls and D-branes: the rules of interaction*, *JHEP* **03** (2005) 019 [hep-th/0501207].

[5] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, *Moduli space of non-abelian vortices*, *Phys. Rev. Lett.* **96** (2006) 161601 [hep-th/0511088]; *Solitons in the Higgs phase: the moduli matrix approach*, *J. Phys.* **A 39** (2006) R315 [hep-th/0602170];
M. Eto et al., *Non-Abelian vortices on cylinder: duality between vortices and walls*, *Phys. Rev.* **D 73** (2006) 085008 [hep-th/0601181];
M. Eto et al., *Non-Abelian vortices of higher winding numbers*, *Phys. Rev.* **D 74** (2006) 065021 [hep-th/0607070];
M. Eto et al., *Constructing non-abelian vortices with arbitrary gauge groups*, *Phys. Lett.* **B 669** (2008) 98 [arXiv:0802.1020].

[6] A. Connes and J. Lott, *Particle models and noncommutative geometry (expanded version)*, *Nucl. Phys.* **18B** *(Proc. Suppl.)* (1991) 29;
A.H. Chamseddine and A. Connes, *The spectral action principle*, *Commun. Math. Phys.* **186** (1997) 731 [hep-th/9606001]; *A universal action formula*, hep-th/9606056.

[7] R. Coquereaux, G. Esposito-Farese and G. Vaillant, *Higgs fields as Yang-Mills fields and discrete symmetries*, *Nucl. Phys.* **B 353** (1991) 689;
R. Coquereaux, G. Esposito-Farese and F. Scheck, *Noncommutative geometry and graded algebras in electroweak interactions*, *Int. J. Mod. Phys.* **A 7** (1992) 6555;
R. Coquereaux, R. Haussling and F. Scheck, *Algebraic connections on parallel universes*, *Int. J. Mod. Phys.* **A 10** (1995) 89 [hep-th/9310148].

[8] K. Morita and Y. Okumura, *Weinberg-Salam theory in noncommutative geometry*, *Prog. Theor. Phys.* **91** (1994) 959; *Reconstruction of Weinberg-Salam theory in noncommutative geometry on $M(4) \times Z(2)$*, *Phys. Rev.* **D 50** (1994) 1016.

[9] J.C. Varilly and J.M. Gracia-Bondia, *Connes' noncommutative differential geometry and the Standard Model*, *J. Geom. Phys.* **12** (1993) 223;
A. Sitarz, *Higgs mass and noncommutative geometry*, *Phys. Lett.* **B 308** (1993) 311 [hep-th/9304005];
D.S. Hwang, C.-Y. Lee and Y. Ne'eman, *BRST quantization of gauge theory in noncommutative geometry: matrix derivative approach*, *J. Math. Phys.* **37** (1996) 3725 [hep-th/9512215].

[10] E. Teo and C. Ting, *Monopoles, vortices and kinks in the framework of non-commutative geometry*, *Phys. Rev.* **D 56** (1997) 2291 [hep-th/9706101].

[11] H.B. Nielsen and P. Olesen, *Vortex-line models for dual strings*, *Nucl. Phys.* **B 61** (1973) 45.

[12] E.J. Weinberg, *Multivortex solutions of the Ginzburg-Landau equations*, *Phys. Rev.* **D 19** (1979) 3008.

[13] N.S. Manton and S.M. Nasir, *Volume of vortex moduli spaces*, *Commun. Math. Phys.* **199** (1999) 591 [hep-th/9807017].

[14] F.A. Schaposnik, *Vortices*, hep-th/0611028.

[15] E. Corrigan, D.B. Fairlie, R.G. Yates and P. Goddard, *The construction of selfdual solutions to* SU(2) *gauge theory*, *Commun. Math. Phys.* **58** (1978) 223.

[16] C.N. Yang, *Condition of selfduality for* SU(2) *gauge fields on euclidean four-dimensional space*, *Phys. Rev. Lett.* **38** (1977) 1377.

[17] O. Lechtenfeld, A.D. Popov and R.J. Szabo, *Quiver Gauge Theory and Noncommutative Vortices*, *Prog. Theor. Phys. Suppl.* **171** (2007) 258 [arXiv:0706.0979];
A.D. Popov, *Integrability of vortex equations on Riemann surfaces*, arXiv:0712.1756;
O. Lechtenfeld, A.D. Popov and R.J. Szabo, *$SU(3)$-equivariant quiver gauge theories and nonabelian vortices*, *JHEP* **08** (2008) 093 [arXiv:0806.2791];
A.D. Popov, *Non-abelian vortices on Riemann surfaces: an integrable case*, *Lett. Math. Phys.* **84** (2008) 139 [arXiv:0801.0808].

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